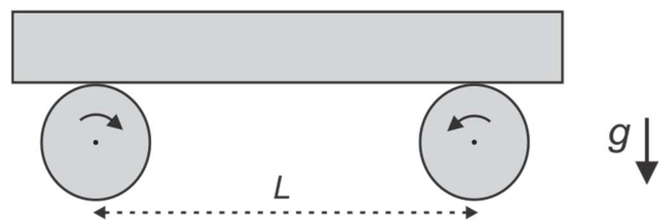
Ph.D. QUALIFYING EXAMINATION – PART A

Tuesday, January 14, 2020, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed.

A1. A square slab of uniform density and total mass M sits horizontally on two parallel cylinders whose centerlines are a distance L apart. The cylinders are fixed and rotate rapidly in opposite directions as shown in the figure. The coefficient of friction between the slab and the cylinders is μ. Initially, the center of the slab is located at a distance a from the midpoint between the cylinders. At time $t = 0$, the slab is released from rest.

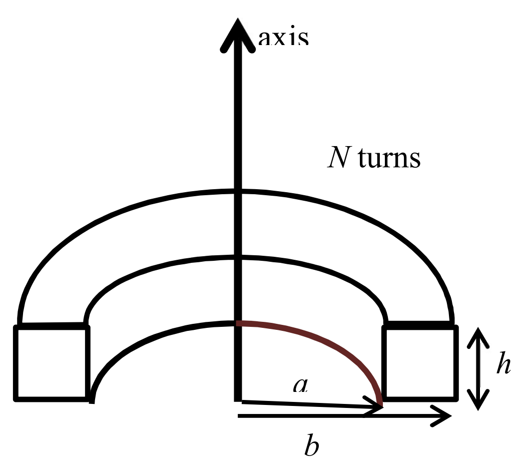


Determine the trajectory of the slab for times $t > 0$.

A2. The figure shows a cut-away side view of a toroidal coil with a rectangular cross section (inner radius a, outer radius b, height h) that carries a total of N closely wound turns.

a) Determine the self-inductance L of the toroidal coil.

A long straight wire runs along the axis of the toroidal coil (like the axle of a wheel). The toroidal coil is connected to a resistor R. The current in the long wire as a function of time is given by $I(t) = I_0 e^{-t/\tau}$, where I_0 and τ are constants.



b) Use Faraday's Law to determine an expression for the emf induced in the toroidal coil and the induced current $I_R(t)$ in the resistor.

c) The induced current $I_R(t)$ in the resistor will cause a back emf in the toroidal coil. Determine an expression for the back emf in the coil due to the induced current $I_R(t)$.

d) What is the ratio of this back emf and the "direct" emf in part (b)?

A3. The Pauli matrices are needed for this problem. They are:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Initially a proton has its spin oriented in the positive x-direction, Thus at time $t = 0$ the spin wave function is $\psi(0) = |x+\rangle$. From time $t = 0$ to time $t = T$ a magnetic field $\vec{B} = B_0\,\hat{z}$ is switched on which will cause the proton to precess in the field. At time T the field is instantly changed to $\vec{B} = B_0\,\hat{y}$ and the spin is again allowed to precess from time $t = T$ to $t = 2T$. The field is then switched off and a measurement of the *x*-component of the spin is measured $\left(\hat{S}_x\right)$. What is the probability of obtaining: $+\hbar/2$?

A4. A relativistic particle of mass m has a Lorentz factor γ. It scatters off an identical particle at rest. After an elastic collision has taken place, it is determined that both particles have the same final energy. In terms of system parameters, find the angle between the directions of motion of the two particles after the collision.

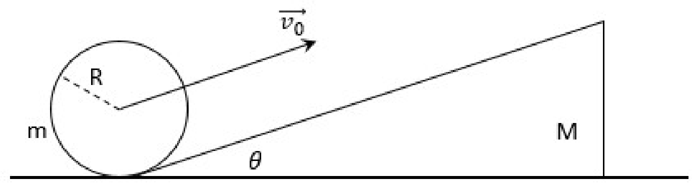
A5. Consider the energy eigenvalue problem $\widehat{H}\,\varphi = \varepsilon\,\varphi$ for a particle moving in 1D in the presence of a "repulsive" one-dimensional delta-function potential $V(x) = \alpha\,\delta(x)$, were α is a positive real constant.

a) Write down the most general form of an acceptable solution to the energy eigenvalue equation for a state of positive energy ε, for (i) $x < 0$, and (ii) $x > 0$; express your solution in terms of the positive constant $k = (2m\varepsilon/\hbar^2)^{1/2}$.

b) Assume continuous solutions and show that, in general, the energy eigenfunction φ has a discontinuity $\Delta\varphi'$ in its derivative at the origin.

c) Construct stationary solutions in which an incident particle approaches the origin from the left with unit amplitude, i.e., $\varphi_0 = e^{ikx}$, and interacts with the potential to generate a scattered wave having a reflected part $\varphi_r = r\,e^{-ikx}$, for $x < 0$, and a transmitted or forward scattered part $\varphi_t = t\,e^{ikx}$, for $x > 0$. Determine the relative reflection and transmission probabilities for such a scattering event. Interpret your results for large and small values of the incident momentum.

A6. A hollow cylinder of mass m and radius R rolls up an inclined plane of angle θ without slipping. The inclined plane has a mass M and is free to slide along the horizontal surface without friction. The cylinder has an initial velocity $\vec{v_0}$ up the inclined plane. The inclined plane is initially at rest with respect to the horizontal surface.



a) After some time the cylinder stops rotating and begins to roll back down the inclined plane. At this moment, what is the horizontal component of velocity of the inclined plane?

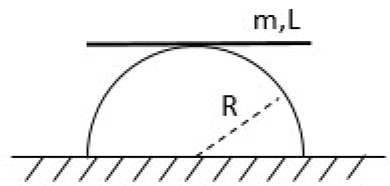
b) How high has the cylinder risen along the incline at this point?

Ph.D. QUALIFYING EXAMINATION – PART B

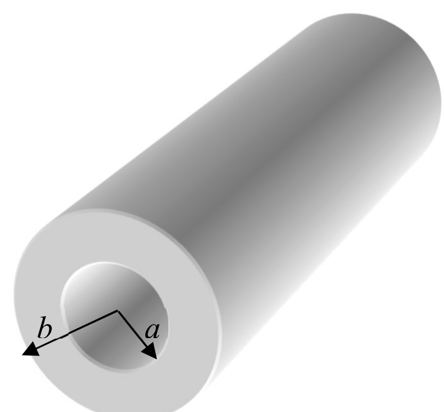
Wednesday, January 15, 2020, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed.

B1. Find the frequency of small oscillations for the thin rod of mass m and length L balanced on top of the **fixed** half-cylinder of radius R.



B2. A current flows down a long hollow straight wire of inner radius a and outer radius b. The hollow wire is made of linear material with magnetic susceptibility χ_m. The free current density is given by $\vec{J}_f = As^2\,\hat{z}$ for $a < s < b$, where A is a constant and the cylindrical coordinates are defined as (s, ϕ, z).



a) Use Ampere's Law in matter to determine the magnetic field $\vec{B}$ in all three regions: inside the cylindrical hole $(s < a)$, inside the cylindrical hollow wire $(a < s < b)$, outside the wire $(s > b)$.

b) Determine the magnetization $\vec{M}$ in all three regions listed above.

c) Determine the volume bound current density $\vec{J}_b$ and the bound surface current $\vec{K}_b$.

d) Calculate the net bound current flowing down the wire.

B3. **Statistical Mechanics: Effusion of ideal gas**

A cubic box of linear size L contains N particles of mass m. The particles can be considered classical non-interacting point masses. The system is in equilibrium at temperature T. A small hole of cross section A is made in one of the walls, causing particles to escape. Find the average energy (per particle) of the escaping particles right after the hole is opened and compare it to the average energy of the particles in the box.

Notes: The system is macroscopic, i.e., N>>1 and A<<L^2. Also, $\int_0^\infty dx\, x\, exp(-ax^2) = 1/(2a)$, $\int_0^\infty dx\, x^3\, exp(-ax^2) = 1/(2a^2)$, and $\int_0^\infty dx\, x^5\, exp(-ax^2) = 1/a^3$.

B4. A free particle moves in 1D along the x-axis. At $t = 0$, it's momentum space wave function is given by the relation

$$\psi(k) = (2\pi)^{-\frac{1}{2}} \int \psi(x)\, e^{-ikx}\, dx = Ae^{-\alpha k^2 - 2\beta k}$$

where $k = p/\hbar$, and $\beta = \beta_0 + i\beta_0$ is a complex constant with real and imaginary parts equal in magnitude.

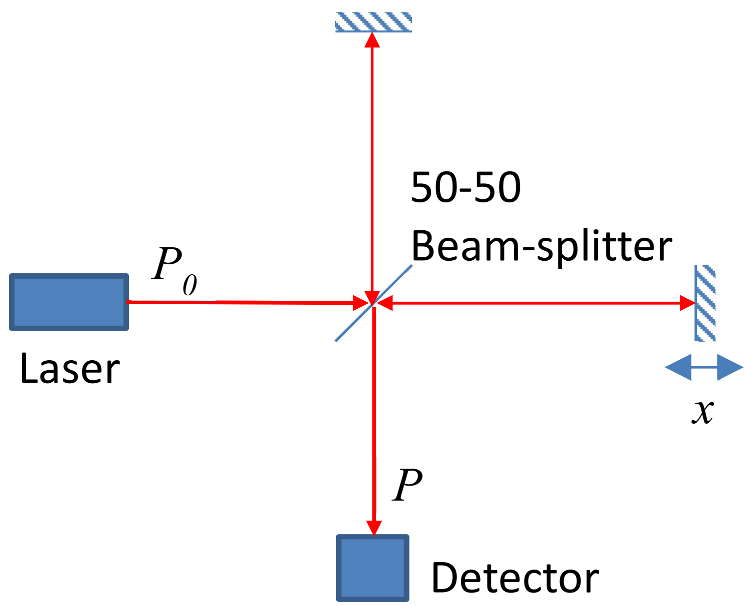
a) If a momentum measurement is made at $t = 0$, what is the most likely value that will be obtained?

b) If instead, a position measurement is made at $t = 0$, what is the most likely value that will be obtained?

c) What is the mean kinetic energy of this particle?

d) Find, up to a normalization constant, the wave function $\psi(x, t)$ for this state for times $t > 0$.

B5. A Michelson interferometer is used to measure small displacements x of one of two mirrors. A laser emitting power P_0 [energy per unit of time] is used as a source. Each photon has energy $\hbar\omega$. After dividing the beam in half by an ideal 50-50 beam-splitter, the individual beams are recombined and the power $P = P_0 \cos^2(kx)$ is registered at the detector, where $k = 2\pi/\lambda$. Because emission of light by the laser is an intrinsically random process, the number of photons $p(n)$ arriving at the detector during a time T is given by the Poisson distribution $p(n) = N^n \frac{e^{-N}}{n!}$, where N is the mean value of n. Therefore, the error δx in determining the position x is due to random deviations of the power P from its average value. Find value(s) of x at which the error (i.e. variance of x) is minimum. Assume that the duration of the measurement is equal to T.



Hint: The error in x is due to the uncertainty in the number of photons registered by the detector. Thus, as a first step, determine the variance of x in terms of the variance of n, the photon number.

B6. Consider a medium of large conductivity and low relative permeability (e.g. a plasma), so that the displacement current can be neglected in Maxwell's equations. Therefore

$$\mathbf{J} = \sigma \mathbf{E} \gg \partial \mathbf{D}/\partial t \text{ and } \mathbf{B} \approx \mu_0 \mathbf{H}$$

a) Show that the magnetic field satisfies the diffusion equation: $\mathrm{D}\nabla^2 \mathbf{B} = \partial \mathbf{B}/\partial t$, where D is the coefficient of diffusion (your derivation will give you an expression for it).

b) Suppose that at time $t = 0$ the magnetic field **B** is uniform inside a sphere of radius R, and directed along the z–axis. Also, there is no field outside (this does not mean $\sigma = 0$ outside). Use the method of separation of variables in spherical coordinates to show that the solution for **B**, assuming $\mathbf{B}(\mathbf{x}, t) = B(r, t)\,\hat{\mathbf{z}}$, must be given by

$$B(r, t) = \frac{2}{r} \int_0^\infty A(k) \sin(kr) \exp(-k^2 \mathrm{D} t)\, dk$$

As usual, it is convenient to write $rB = U(r)T(t)$ and use $\nabla^2 B = (1/r) d^2/dr^2 (rB)$. Note that you need only derive the *r*-dependence and *t*-dependence, as well as explain the needed integration and its range. You do **not** need to derive the explicit form of $A(k)$.

c) $A(k)$ can be obtained from the initial condition. It can then be shown that over a time $\tau = R^2/\mathrm{D}$ the field rapidly drops to about a tenth of its value, and $B \propto (R^2/\mathrm{D}t)^{3/2}$ afterwards. Assuming the sphere represents the core of the Earth ($R \approx 3{,}000$ km, $\sigma \approx 10^5/\Omega\mathrm{m}$), how long (*i.e.* τ in years) would it take for the Earth to lose its magnetic field? A year is approximately $3.2 \cdot 10^7$s. (This shows that the field must be continuously regenerated, most likely by fluid motion in the outer core)